# Cause of Stock Return Stochastic Volatility: Query by Way of Stochastic Calculus

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**Abstract.** This study uses stochastic calculus to investigate the causes of the stock return stochastic volatility. The study aims to advance new explanations of stochastic volatility that hold also if the firm is unleveraged, and if the level of uncertainty about future business conditions does not change. Using the dividend discount model, I show that stock return volatility is admittedly stochastic if future dividends are affected by more than one stochastic state variable. Morever, I study how the mappings of state variables are related to stochastic return volatility. This study also investigates the effects of the discount rate and state variables' mutual correlation on the level of stock volatility and its fluctuation, finding substantial relationships therein.

**Keywords:** Stochastic calculus, Stochastic volatility, Geometric Brownian motion, Dividend discount model.

## 1 Introduction

It is by now a widely accepted observation that the volatilities of individual stocks and aggregate stock markets are not constants, but change stochastically over time. The literature has repeatedly presented sophisticated statistical models to describe stochastic volatility, but the question of *why* stock return volatility varies remains open. [Schwert, 1989] presents an extensive analysis of the relation between market volatility and economic activity, confirming Officer's [Officer, 1973] earlier results that market volatility is higher during economic downturns. This has also been justified by [Hamilton and Lin, 1996], who found that economic recessions are the single most important factor explaining market volatility, accounting for about 60 percent of its variation. Further, changes in volatility are also related to financial and operating leverage, personal leverage, interest rates, inflation rates, money growth, industrial production growth, trading volume, trading halt, and program trading (see, for example, [Black, 1976], [Christie, 1982], [Mascaro and Meltzer, 1983], [Schwert, 1989], [Schwert, 1990]). Stock volatility is also stochastic if a stock has option characteristics; that is, the stock can be viewed as an option on the leveraged firm's assets (see [Merton, 1974]), and the firm may have numerous growth options. Overall in the literature, the usual explanation of stochastic volatility is either corporate leverage or a change in the level of uncertainty about future macroeconomic conditions.

This study advances new explanations of stochastic volatility by way of stochastic calculus. My explanation does not challenge the earlier explanations but rather complements them.

This paper investigates the significance of the above observation by suppressing the other possible causes of stochastic volatility. The assumption of risk-neutrality suppresses leverage effects, the assumption of a constant interest rules out the possibility that the randomness of stock volatility is driven by varying interest rates, and the assumption of constant state variable volatilities eliminates any change in the uncertainty about future macroeconomic conditions. I also examine the effects of state variables' mutual correlations and the discount rate on the level of volatility and its variation.

## 2 Model

To suppress the leverage effect, we assume that the risk-neutrality, and hence the discount rate, denoted by $\mu$, equals the risk-free interest rate. Moreover, by assuming a constant interest, we rule out the possibility that the randomness of volatility is driven by varying interest rates. The stochasticity of stock volatility arise solely from the dividend process. We denote the dividend stream by $\{D_{t_1}, D_{t_2}, \ldots, D_{t_n}\}$, where dividends occur at (known) times $t_1, t_2, \ldots, t_n$. The stock price is assumed to equal the cumulative present value of its expected dividends,

$$V(t) = \sum_{k=1}^{n} \mathbb{E}_t \left[ D_{t_k} \right] \exp\left[ -\mu(t_k - t) \right], \tag{1}$$

Suppose that a discrete dividend stream consists of $n$ discrete dividends. All investors are assumed to monitor the processes of state variables and continuously revising their beliefs regarding to expected dividends. Each dividend is driven by an $m$ stochastic state variable. In addition, the dividends' state variable vectors need not be equal; that is, the next year's dividend can be driven by state variables different from those of the dividend paid after five years. Consequently, we specify the dividend process in rather general terms and allow that, for example, each dividend depends on the interest rate and inflation, but that the two first dividends also depend on the oil price, whereas later dividends depend on the price of biodiesel instead of oil. Overall, the matrix $\mathbf{X}(t) \in \mathbb{R}^{m\times n}$, represents the dividend stream information apprearing at time $t$, and vector $\mathbf{X}_k(t)$ represents the information associated with the stock dividend $k$ (that will be paid at time $t_k$).

For all $k = 1 \ldots n$, $D_{t_k} : \mathbb{R}^m \mapsto \mathbb{R}_+$ is a known mapping, and $\{\mathbf{X}_k(t); t \geq 0\}$, $\mathbf{X}_k \in \mathbb{R}^m$ the state variable vector of the dividend $D_{t_k}$, is a linear diffusion defined on a complete filtered probability space $(\Omega, \mathcal{F}^{X_k}, \mathcal{P})$. We assume that for all $i = 1 \ldots m, k = 1 \ldots n$, $\{X_{ik}(t); t \geq 0\}$ evolves according to the stochastic differential equation

$$dX_{ik}(t) = a_{ik}(t, X_{ik})dt + b_{ik}(t, X_{ik})dW_{ik}(t), X_{ik}(0) := x_{ik}, \tag{2}$$

where $W_{ik}$ is a standard Wiener process with the instantaneous correlations $dW_{ik}dW_{jl} = \rho_{ik,jl}dt$ for all $i, j = 1 \ldots m$, $k, l = 1 \ldots n$. Read the above such that $X_{ik}$ is the state variable $i$ of the dividend $k$ (a dividend that will be paid at time $t_k$).

According to Itô's lemma, the stock price must itself follow the Itô process:

$$dV = \alpha(t)V dt + \sum_{i,k} b_{ik} \frac{\partial V}{\partial x_{ik}} dW_{ik}, \tag{3}$$

where $i, j = 1 \ldots m$, $k, l = 1 \ldots n$, and where $\alpha$ is the expected price appreciation. The term $\sum_{i,k} b_{ik} \frac{\partial V}{\partial x_{ik}} dW_{ik}$ makes the stock price behave stochastically. The rest of the study focuses on just this term.

## 3 Causes of Stochastic Return Volatility

### 3.1 Mappings of State Variable

For simplicity the analysis, let us suppose temporarily that there is only one stochastic state variable driving all the dividends, and that the state variable of the dividends $\{X(t); t \geq 0\}$ itself follows the geometric Brownian motion:

$$dX(t) = \theta X(t)dt + \sigma X(t)dW(t), \ x > 0, \tag{4}$$

where $\theta \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ are constants. We will relax assumptions later when we will consider multiple state variables and the alternative processes of state variables. Now the asset price process is in the form

$$\frac{dV}{V} = \alpha dt + \sigma \frac{x}{V} \frac{\partial V}{\partial x} dW = \alpha dt + \sigma \varepsilon dW,$$

where $\alpha$ equals the expected price appreciation and $\varepsilon$ is the elasticity of the asset price to the state variable. If $\varepsilon$ is stochastic, so is stock return volatility $\sigma\varepsilon$, too.

**Proposition 1.** *Suppose that dividends' state variable $x$ follows geometric Brownian motion. If the elasticity of the asset price to the state variable is a constant, say $\zeta$, the asset price $V = cx^\zeta$, where $c$ is a positive constant.*

*Proof.*
The solution of the differential equation $\frac{\partial V}{\partial x}\frac{x}{V} = \zeta$ is $V = cx^\zeta$, where $c$ is a positive constant. □

*Remark 1.* The above proposition also says that if the asset price is not in the form of $V = cx^\zeta$, then the elasticity of the asset price to the state variable, and hence also stock return volatility, is not constant.

*Example 1.* The above means that the assumption of constant return volatility is invalid, if, for example, the asset price $V = c_1x^b + c_2$, where $b, c_1, c_2$ are constants. This is the case if the dividend of time $t_k$,

$$D_{t_k} = a_1 X(t_k)^b + a_2. \tag{5}$$

If $x$ follows (4), then Now $\varepsilon$ is admittedly stochastic:

$$\frac{\partial V}{\partial x}\frac{x}{V} = \frac{c_1 x^b b}{c_1 x^b + c_2},$$

where $c_1 = a_1 \sum_{k=1}^{n} e^{(\theta b + \frac{1}{2}b(b-1))(t_k - t)}$ and $c_2 = a_2c_1/a_1$. □

*Remark 2.* Example 2 has an important economic interpretation. Suppose that the price of output good fluctuates according to (4) and that the dividend of time $t_k$ is $X(t_k) - a$, where $a$ denotes the production costs of the period. Then return volatility equals $\sigma \frac{x}{x-a}$ and is stochastic.

*Example 2.* Another example of stochastic return volatility could be $V = f_1(x) + f_2(x) + \ldots$, where $f_i$, $i = 1, 2, \ldots$, is the function of $x$. This is a case if, for example, the dividend of time $t_k$, $D_{t_k} = \ln X(t_k)$, when $\varepsilon$ is stochastic even if discount rate $\mu$ equals zero and number of dividends $n$ equals one:

$$\frac{\partial V}{\partial x}\frac{x}{V} = \frac{1}{\ln X(t) + (\theta - \frac{1}{2}\sigma^2)(t_k - t)}.$$

□

We have now considered transformations that result in return volatility to be stochastic. What kind of dividend transformations do produce constant return volatility? The next proposition clarifies this.

**Proposition 2.** *Suppose that dividends' state variable $x$ follows geometric Brownian motion. If the dividend of time $t$ has a form $D_{t_k} = aX(t_k)^b$, where $a$ and $b$ are constant numbers, then the elasticity of the asset price to the state variable, $\varepsilon$, is constant and equals $b$.*

*Proof.* The proposition is a direct implication of Example 2 with $a_2 = 0$ in (5). □

### 3.2 Multiple State Variables

We assume that

$$D_{t_k} = D(\mathbf{X}_k(t_k)) = X_{1k}(t_k) + X_{2k}(t_k) + \cdots + X_{mk}(t_k), \tag{6}$$

$k = 1 \ldots n$, and that the state variables of the dividends $\{\mathbf{X}(t); t \geq 0\}$ follow the geometric Brownian motion with *constant* drifts and volatilities :

$$dX_{ik}(t) = \theta_{ik} X_{ik}(t)dt + \sigma_{ik} X_{ik}(t) dW_{ik}(t), \tag{7}$$

where $x_{ik} > 0$ for all $i, k$. Now stock price as a cumulative present value of expected dividends can be expressed as

$$V(t) = \sum_{i,k} X_{ik}(t) \exp\left((\theta_{ik} - \mu)(t_k - t)\right), \tag{8}$$

where $i = 1 \dots m, k = 1 \dots n$, and where $\mu$ is the stock's instantaneous discount rate.

The stochastic term in (3) takes the form

$$\sum_{i,k} b_{ik} \frac{\partial V}{\partial x_{ik}} dW_{ik} = \sum_{i,k} \sigma_{ik} X_{ik}(t) \exp\left((\theta_{ik} - \mu)(t_k - t)\right) dW_{ik}(t), \tag{9}$$

$i = 1 \dots m, k = 1 \dots n$. How should we interpret this? What can we say now about stock volatility?

**Proposition 3.** *Suppose that there are several (more than one) state variables and that (6), (7), and (8) hold. Then the stock return volatility is a constant if and only if all the state variables are drived by perferctly correlated Wiener processes and volatilities of state variables, $\sigma_{ik}$, are equal to each other.*

*Proof.* Without loss of generality, suppose that all dividends depend on the same two state variables, $X_1$ and $X_2$. Technically, for all $k = 1 \dots n$

$$D(X_1(t_k), X_2(t_k)) = X_1(t_k) + X_2(t_k).$$

These variables follows the geometric Brownian motions driven by the Wiener processes $W_1(t)$ and $W_2(t)$, $dW_1(t)dW_2(t) = \rho_{12}dt$. Let $W^*$ be an independent Wiener process with respect to $W_1$ and $W_2$. Then we can write $dW_2(t) = \rho_{12}dW_1(t) + \sqrt{1-\rho_{12}^2}dW^*(t)$. If $\sigma_{ik} = \sigma$ and $\rho_{12} = 1$, then we can then write (9) as follows:

$$\begin{aligned}\sum_{i,k} b_{ik} \frac{\partial V}{\partial x_{ik}} dW &= \left\{ \sum_{i,k} X_{ik}(t) \exp\left((\theta_{ik} - \mu)(t_k - t)\right) \right\} \sigma dW(t) \\ &= V \sigma dW(t),\end{aligned}$$

where $dW_1(t) = dW_2(t) = dW(t)$. Thus, the stock price evolves with constant volatility. We must still prove that if return volatility is constant, then all the state variables are drived by perferctly correlated Wiener processes and volatilities of state variables, $\sigma_{ik}$, are equal to each other. Let us do this by showing that if processes are not perefectly correlated or if volatilities are not the equal, return volatility is not constant. Again, suppose that all dividends depend on the same two state variables, $X_1$ and $X_2$. The price of the stock is

$$V(t) = \sum_{k=1}^{n} \left\{ X_1(t) \exp((\theta_1 - \mu)(t_k - t)) + X_2(t) \exp((\theta_2 - \mu)(t_k - t)) \right\}. \tag{10}$$

We can then write (9) as follows:

$$\sum_{i,k} b_{ik} \frac{\partial V}{\partial x_{ik}} dW_{ik} = \phi_1(t)V(t)dW_1(t) + \phi_2(t)V(t)dW_2(t),$$

where

$$\phi_i(t) \equiv \frac{\sigma_i X_i(t) \sum_{k=1}^{n} \exp((\theta_i - \mu)(t_k - t))}{V(t)},$$

$i = 1, 2$. Let $\tilde{\phi}(t)d\tilde{W}(t) = \phi_1(t)dW_1(t) + \phi_2(t)dW_2(t)$, when we find that (see, for example, [Nielsen, 1999, pp. 25, 75-76])

$$\tilde{\phi}(t) = \sqrt{\phi_1(t)^2 + \phi_2(t)^2 + 2\rho_{12}\phi_1(t)\phi_2(t)}. \tag{11}$$

Therefore, stock price diffusion takes the form

$$dV(t) = \alpha V(t) + \tilde{\phi}(t)V(t)d\tilde{W}(t). \tag{12}$$

Clearly, the stock volatility $\tilde{\phi}$ is admittedly stochastic even if either $\sigma_1 = \sigma_2$ or $\rho_{12} = 0$. Note that if return volatility depends positively on the correlation $\rho_{12}$. □

I have illustrated the stochastic volatility numerically by generating sample paths for state variablesand supposing that equations (7), (10) and (11) determine state variables, stock price, and volatility. The illustration can be found at
`http://www.tut.fi/~kanniain/ASMDA/illustration.pdf`
I assumed two state variables, $X_1(t)$ and $X_2(t)$ with a constant correlation and with constant volatilities and drifts. Moreover, I assumed that $n$ dividends will be paid at times (years) $1, 2, 3 \ldots$, and I simulated the time interval $(0, 1)$ (the first year). The illustration shows that stock return volatility may vary considerably over time. Moreover, it also examines the effect of the correlation $\rho_{12}$ on the price paths concluding that if the correlation increases, the stock price becomes more volatile, and the volatility curve moves upward. If the correlation decreases, also stock volatility decreases, and the volatility curve drops. This is in line with our analytical observation of a positive relation between $\rho_{12}$ and $\tilde{\phi}$. The result is analogous with portfolio diversification: if state variables do not correlate mutually, their fluctuations eliminate each other. Moreover, the numerical illustration argues that the greater (less) the correlation, the greater (less) the stock volatility but with less (greater) fluctuation. The result is also quite intuitive. As we can see from the above equations, the mutual proportions of the state variable values clearly affect the level of stock volatility. Suppose that dividends depend on two variables, and that the volatility of the first variable, $\sigma_1$, is less than the volatility of the second variable, $\sigma_2$. If the value of the second state variable,

$X_2$, increases in proportion to the value of the second state variable, $X_1$, the more volatile state variable takes room, and stock volatility increases. Similarly, if $X_1$ increases in proportion to $X_2$, stock volatility decreases. Obviously also the greater (less) the correlation between state variables, the less (greater) their mutual proportion changes over time. Therefore, if state variables evolve with different volatilities and a low, or even negative mutual correlation, stock volatility may fluctuate substantially. We could interpret this result economically in that if a business depends on homogenous (heterogeneous) factors (in the sense of statistical dependency), its volatility is high (low) and does not (does) vary much. I also illustrated the effect of discount rate to volatility and its fluctuation. Here the effect of the discount rate on stock volatility can be either positive or negative. In addition, the discount rate has a great effect on how stock volatility varies over time.

### 3.3 Alternative Characterizations of State Variables Processes

Finally, we consider alternative processes of the state variables. Suppose that $a_{ik}(t, X_{ik}) = \eta_{ik}\left(\bar{X}_{ik} - X_{ik}(t)\right)$ and $b_{ik}(t, X_{ik}) = \sigma_{ik}$, in which case the state variable $\{X_{ik}(t); t \geq 0\}$, $i = 1 \ldots m, k = 1 \ldots n$ evolves according to the stochastic differential

$$dX_{ik}(t) = \eta_{ik}\left(\bar{X}_{ik} - X_{ik}(t)\right)dt + \sigma_{ik}dW_{ik}(t),$$

where $\eta_{ik}$, $\bar{X}_{ik}$, and $\sigma_{ik}$ are constant numbers for all $i, k$. This is the so-called mean-reverting process. Now, because for some $\tau > t$ (see, for example, [Dixit and Pindyck, 1994, p. 74])

$$\mathbb{E}_t\left[X_{ik}(\tau)\right] = \bar{X}_{ik} + \left(X_{ik}(t) - \bar{X}_{ik}\right)\exp\left(-\eta_{ik}(\tau - t)\right),$$

the stock price with mapping $D_{t_k} = X_{1k}(t_k) + X_{2k}(t_k) + \cdots + X_{mk}(t_k)$

$$V(t) = \sum_{i,k}\left\{\bar{X}_{ik} + \left(X_{ik}(t) - \bar{X}_{ik}\right)\exp\left(-\eta_{ik}(t_k - t)\right)\right\}\exp\left(-\mu(t_k - t)\right).$$

Stock return volatility is now unquestionably stochastic since the stochastic term in (3) takes the form

$$\sum_{i,k} b_{ik}\frac{\partial V}{\partial x_{ik}}dW_{ik}(t) = \sum_{i,k}\sigma_{ik}\exp\left(-(\eta_{ik} + \mu)(t_k - t)\right)dW_{ik}(t).$$

Note that stock volatility would be stochastic even if $\sigma_{ik} = \sigma$ and $W_{ik}(t) = W(t)$ for all $i = 1 \ldots m, k = 1 \ldots n$, $t > 0$, in which case volatility would be equal to

$$\frac{\sigma}{V}\sum_{i,k}\exp\left(-(\eta_{ik} + \mu)(t_k - t)\right).$$

*Remark 3.* The reason here is that the variance rate does not grow with $x$. Therefore, if dividends are driven by such a process, stock volatility remains unquestionably stochastic with linear mappings of state variables.

## 4 Conclusions

The starting point of this study was that dividends are driven by state variables and that investors monitor the state variables and continuously revise their beliefs regarding to stock price. The paper studied how the mappings of state variables are related stochastic return volatility. Moreover, also multiple state variables were explored. The main result is that the stock return volatility is admittedly stochastic if future dividends are affected by more than one stochastic state variable. Thus, the paper affirms the invalidity of the geometric Brownian motion as models of stock price. We observed that the correlation between state variables has an effect on volatility dynamics according to the greater (less) the correlation, the greater (less) the stock volatility, with less (more) fluctuation over time. In addition, we found that the discount rate affects volatility and its fluctuation positively or negatively.

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