On a numerical approximation method of evaluating the interval transition probabilities of semi-Markov models

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Abstract. For the classical semi-Markov model, either time homogeneous or non-homogeneous, an examination of the convergence of the interval transition probabilities $p_{ij}(s,t)$ as $t\rightarrow\infty$ is presented using an approximation method provided by [R. De Dominics and R. Manca 1984]. Especially, we examined for various values of the step $h$ the dependence of the accuracy of the respective numerical method in finding the transition interval probabilities and we investigated the complexity of this algorithm.

Keywords: Semi-Markov process, Non homogeneity, Numerical methods.

1. Introduction

In what follows denote by $E=\{1,2,...,n\}$ the state space of the Markov model and by $t$ the (continuous-) time parameter. The states $1,2,...,n$ are exclusive and exhaustive, i.e., every member of the system may be in one and only one state at some time point $t$. Also denote by $P_{ij}$, $i,j=1,2,...,n$ the probability that a member who entered state $i$ on its last transition will enter state $j$ on its next transition. The transitions between the states occur according to either a homogeneous or a non-homogeneous semi-Markov chain (the embedded semi-Markov chain). Let $P=\{P_{ij}\}$ be the matrix of transition probabilities and $G(s,t)=\{G_{ij}(s,t)\}$ the matrix of the distributions of the holding times, where:

$G_{ij}(s,t)=$prob{the holding time in state $i$ is less than or equal to $t-s$, given that the entrance in $i$ occurred at time $s$ and next transition will take place in state $j$}.

The core matrix of the semi-Markov process is $Q(s,t) = \{Q_{ij}(s,t)\}$, with
where the symbol “◊” stands for the Hadamard product. Finally, we denote $S(s,t)=\text{diag}\{S_i(s,t)\}$, where:

$$S_i(s,t)=\sum_{j=0}^{\infty} Q_{ij}(s,t) = \text{prob}\{ \text{the holding time in state } i \text{ is less than or equal to } t-s, \text{ given that the entrance in } i \text{ occurred at time } s \}$$

The aspects which have been studied in discrete and also in continuous time semi-Markov models as outlined above, are the asymptotic behaviour, stability, variability, attainability etc. Semi Markov models arise in physics, actuarial work, biometry and manpower planning. Basic results can be found in [McClean, 1978,1980,1986], [Mehlman, 1979], [Bartholomew, 1986], [Howard, 1971], [Janssen and Limnios, 1999], [R. De Dominics and R. Manca, 1984], [Papadopoulos and Vassiliou, 1994] and [Vassiliou and Papadopoulou, 1992].

The most characteristic features of a semi-Markov model are related to the interval transition probabilities of the model. A recursive formula for the interval transition probabilities is

$$P_i(s,t)=\delta_{ij}(1-S_i(s,t)) + \sum_{k}^t P_j(u,t)d(Q_{ik}(s,u)). \quad (1)$$

Relation (1) can be approximated numerically by the recursive relation

$$P_{i,j}(s,t)=D_{i,j}(s,t) + \sum_{h=s}^{t} \sum_{k=0}^{h} V_{i,j,k}(s,t)P_{i,j}(s,t), \quad (2)$$

(R. De Dominics and R. Manca (1984)), where

- The step of the approximation is equal to 1, i.e., the calculations in (2) are carried out only for $h = s, s+1, s+2, \ldots t$
- $D_{i,j}(s,t) = \delta_{ij}(1-S_{i,j}(s,t))$ is the approximation of the probability that the process holds in $i$ without any transition in the time interval $(s,t)$
- $S_{i,j}(s,t) = \sum_{j} Q_{i,j}(s,t)$ is the approximation of the probability that the chain moves from state $i$ into any other state before time $t$, given that the entrance in $i$ took place at time $s$
- $Q_{i,j}(s,t) = G_{i,j}(s,t)P_{i,j}$ is the approximation of the $ij$ element of the core matrix of the semi-Markov process
- $V_{i,j,k}(s,t)$ is the difference $Q_{i,j,k}(s,t) - Q_{i,j,k}(s,t-1)$ for the time interval $(h-1,h)$ if $h \leq s+1$, which interprets an approximation of the p.d.f. $c_{ij}(s,t)$
- $V_{i,j,k}(s,t) = 0$ if $h = s+1$, and $V_{i,j,k}(s,t) = 0$. The recursive formula (2) in matrix notation becomes
Numerical approximation of Semi Markov models

\[ P_{x,t} = D_{x,t} + \sum_{h=1}^{t} V_{x,t} P_{h} , \]  

(3)

with initial conditions \( P_{x,t} = D_{x,t} = I \) and \( V_{x,t} = 0 \), where the \( i,j \)-th element of \( P_{x,t} \) equals \( P_{x,i,j} \), the \( i,j \)-th element of \( D_{x,t} \) equals \( D_{x,i,j} \), and the \( i,j \)-th element of \( V_{x,t} \) equals \( V_{i,j} \).

It is interesting to investigate the convergence of (3) numerically. After some manipulations formula (3) can be written in the form \( UP = D \), where

- \( U \) is an upper block triangular matrix with the \((i,i)\) block equal to the unity matrix and the \((i,j)\)-th block \((i\neq j)\) equal to \(-V_{i,j}\).
- \( P \) is a block upper triangular matrix with the \((i,i)\)-th block equal to the unity matrix and the \((i,j)\)-th block \((i\neq j)\) equal to \( P_{i,j} \).
- \( D \) is a block triangular matrix with the \((i,i)\)-th block equal to the unity matrix and the \((i,j)\)-th block \((i\neq j)\) equal to \( D_{i,j} \).

We examined for various values of the step \( h \) the dependence of the accuracy of the numerical method based on (3) in finding the transition interval probabilities. We examined both the cases where the holding times depend on the time of entrance into some state or not (non-homogeneous and homogeneous case respectively). Finally we investigated the complexity of the algorithm provided by (3).

References


4 D. Bitziadis, G. Tsaklidis, and A. Papadopoulou


Numerical approximation of Semi Markov models 5